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ON A CONJECTURE OF IGLEHART

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On a conjecture of Iglehart *)

by

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Abstract. This paper gives an elementary proof of Iglehart's conjecture about the classical dynamic inventory model with a positive continuous demand. This conjecture states that the minimal total expected cost for a planning horizon of n periods minus n times the minimal long-run average expected cost per period has a finite limit as $n\to\infty$ for each initial stock.

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*) This paper is not for review; it is meant for publication in a journal.

1. *Introduction*

In a fundamental paper Iglehart[4] conjectured for the dynamic inventory model with a linear purchase cost, a fixed set-up cost and convex holding and shortage costs that the minimal total expected cost for a planning horizon of n periods minus n times the minimal long-run average expected cost has a finite limit as $n\to\infty$ for each initial stock. In [1] this conjecture was proved amongst other results for the case of a positive discrete demand by using results in [2]. In this paper we present an elementary proof of the original conjecture offered for the case of a positive continuous demand. The proof applies equally well to the discrete demand case.

In section 2 we formulate the model and give some preliminaries. Also, we state in section 2 the main theorem that will be proved in section 3.

2. *Model and preliminaries*

We consider the single-item inventory model in which the demands in successive periods form a sequence of independent random variables having a common probability distribution with density $\phi(\cdot)$. It is assumed that $\phi(\xi)$ is positive for all ξ sufficiently large. Further we suppose that the demand per period has a finite expectation μ. Any unfilled demand in a period is backlogged. Hence the stock level may take on any real value, where a negative value indicates the existence of a backlog. At the beginning of each period the stock on hand is reviewed. At each review an order may be placed for any positive amount of stock. An order, when placed, is delivered instantaneously. The demand in each period takes place after review and delivery(if any). The following costs are involved. The cost of ordering an amount of z is $K\delta(z)+cz$, where $K\geq 0$, $c\geq 0$, $\delta(0)=0$, and, $\delta(z)=1$ for $z>0$. Let $L(y)$ be the expected holding and shortage costs in a period when y is the amount of stock at the beginning of that period just after any additions to stock. We assume that $L(y)$ is a nonnegative convex function that is continuous for all y. Further it is assumed that both $L(y)\to\infty$ and $cy+L(y)\to\infty$ as $|y|\to\infty$. Finally, future costs are not discounted.

We now give some known results for this model that will be needed in the sequel. For any real x, let $f_0(x)=0$. It was proved by Scarf[5](see also [3]) that there is a sequence $\{f_n(\cdot),n\geq 1\}$ of continuous functions satisfying, for all x and all $n\geq 1$,

(1) $$f_n(x) = \min_{y\geq x} \{c.(y-x) + K\delta(y-x) + L(y) + \int_0^\infty f_{n-1}(y-\xi)\phi(\xi)d\xi\},$$

such that, for all $n\geq 1$,

(2) $$\begin{aligned} f_n(x) &= -cx + K + G_n(S_n) && \text{for } x<s_n, \\ &= -cx + G_n(x) && \text{for } x\geq s_n, \end{aligned}$$

where $G_n(y)=cy + L(y) + \int_0^\infty f_{n-1}(y-\xi)\phi(\xi)d\xi$, S_n is the smallest number that minimizes the function $G_n(y)$, and s_n is the smallest number less than or equal to S_n for which $G_n(s_n)=K + G_n(S_n)$. Hence the right side of (1) is minimal for $y=S_n$ when $x<s_n$ and for $y=x$ when $x\geq s_n$. It was proved in [3] that the sequences $\{s_n\}$ and $\{S_n\}$ are bounded. Observe that $f_n(x)$ denotes the minimal total expected cost for a planning horizon of n periods when the initial stock is x. Consider now the infinite period model. Denote by $a(s,S)$ the average expected cost per period, see [4]. Fix two finite numbers s^* and S^* such that $a(s^*,S^*)=g$ and $L(s^*) + c\mu=g$ where $g=\min_{s,S} a(s,S)$. In [4] it was shown that such numbers exist and that the (s^*,S^*) policy is average cost optimal among the class of all possible policies. Hence the minimal average expected cost per period is independent of the initial stock and

equals g. Next define the function $\psi(\cdot)$ by

$$\psi(x) = -c.(x-s^*) \qquad \text{for } x<s^*, \tag{3}$$
$$= L(x) - g + \int_0^\infty \psi(x-\xi)\phi(\xi)d\xi \qquad \text{for } x\geq s^*.$$

The relation (3) constitutes for $x\geq s^*$ a renewal equation. Using this and the relation $L(s^*) + c\mu=g$ it is easy to verify that (3) has a unique finite solution $\psi(x)$ which is continuous for all x. It was proved in [4] that, for all x,

$$g + \psi(x) = \min_{y\geq x} \{c.(y-x) + K\delta(y-x) + L(y) + \int_0^\infty \psi(y-\xi)\phi(\xi)d\xi\}, \tag{4}$$

where the right side of (4) is minimal for $y=S^*$ when $x<s^*$ and for $y=x$ when $x\geq s^*$.

In the next section we prove

THEOREM 1. *The sequence* $\{f_n(x) - ng - \psi(x), n\geq 1\}$ *has a finite limit for all* x. *Moreover, the limit is independent of* x.

Iglehart[4] proved this result for the case of K=0 and offered it as a conjecture for the case of K>0.

3. *The proof*

To prove Theorem 1, we fix two finite numbers L and U such that $L<s_n\leq S_n\leq U$ for all $n\geq 1$ and $L<s^*\leq S^*\leq U$. Let $X=\{x|x\leq U\}$, and let $A(x)=\{y|y\geq L, x\leq y\leq U\}$ for $x\in X$. By the results in section 2 we have, for all $x\in X$ and $n\geq 1$,

$$f_n(x) = \min_{y\in A(x)} \{c.(y-x) + K\delta(y-x) + L(y) + \int_0^\infty f_{n-1}(y-\xi)\phi(\xi)d\xi\}, \tag{5}$$

and, for all $x\in X$,

$$g+\psi(x) = \min_{y\in A(x)} \{c.(y-x) + K\delta(y-x) + L(y) + \int_0^\infty \psi(y-\xi)\phi(\xi)d\xi\}. \tag{6}$$

Observe that, by (2) and (3), the above integrals converge absolutely. Define π by $\pi(x)=S^*$ for $x<s^*$ and $\pi(x)=x$ for $x\geq s^*$ and, for $n\geq 1$, define π_n by

$\pi_n(x)=S_n$ for $x<s_n$ and $\pi_n(x)=x$ for $x\geq s_n$. Then, for any $x\epsilon X$, $\pi(x)$ minimizes the right side of (6) and $\pi_n(x)$ minimizes the right side of (5).

For any $x\epsilon X$ and $n\geq 1$, let $e_n(x) = f_n(x) - ng - \psi(x)$. Using the definitions of π and π_n, if follows from (5) and (6) that, for all $x\epsilon X$ and $n\geq 1$,

$$(7) \qquad e_{n+1}(x) \leq \int_0^\infty e_n(\pi(x)-\xi)\phi(\xi)d\xi \text{ and } e_{n+1}(x) \geq \int_0^\infty e_n(\pi_n(x)-\xi)\phi(\xi)d\xi.$$

Since U can be chosen arbitrarily large, Theorem 1 is an immediate consequence of the following Theorem.

THEOREM 2. *The sequence* $\{e_n(x), n\geq 1\}$ *has a finite limit for all* $x\epsilon X$. *Moreover, the limit is independent of* $x\epsilon X$.

PROOF. Using the continuity of $f_1(\cdot)$ and $\psi(\cdot)$, it follows from (2) and (3) that there is a finite number N such that $|e_1(x)|\leq N$ for all $x\epsilon X$. By induction we have from (7) that $|e_n(x)|\leq N$ for all $x\epsilon X$ and $n\geq 1$.

Now, define $M_n(x) = \sup_{x\epsilon X} e_n(x)$ and define $m_n = \inf_{x\epsilon X} e_n(x)$ for $n\geq 1$. By induction it follows from (7) that $M_{n+1}\leq M_n$ and $m_{n+1}\geq m_n$ for all $n\geq 1$. Hence the bounded sequences $\{M_n\}$ and $\{m_n\}$ have finite limits M and m, respectively.

Let $a = \int_0^{U-L} \phi(\xi)d\xi$. Then $a<1$ since $\phi(\xi)>0$ for all ξ sufficiently large. By (2) and (3), $e_n(x) = \varepsilon_n$ for all $x<L$ and $n\geq 1$ where $\varepsilon_n = K + G_n(S_n) - ng - cs.^*$ Since $L\leq\pi(x)\leq U$ for all $x\epsilon X$, we get from the first part of (7) that $e_{n+1}(x) \leq aM_n + (1-a)\varepsilon_n$ for all $x\epsilon X$ and $n\geq 1$. Hence $M_{n+1}\leq aM_n + (1-a)\varepsilon_n$ for all $n\geq 1$. Similarly, we derive from the second part of (7) that $m_{n+1}\geq am_n + (1-a)\varepsilon_n$ for all $n\geq 1$. Hence $M_{n+1} - m_{n+1}\leq a(M_n - m_n)$ for all $n\geq 1$, so, $M - m\leq a(M - m)$. Since $a<1$ and $M\geq m$, we have $M=m$. From $M=m$ and $m_n\leq e_n(x)\leq M_n$ for all $x\epsilon X$ and $n\geq 1$, it follows that the sequence $\{e_n(x)\}$ has limit m for all $x\epsilon X$ as was to be proved.

REMARK. Since $M_n-m_n\leq a^{n-1}(M_1-m_1)$ for $n\geq 1$, the convergence of $e_n(x)$ for $n\to\infty$ is exponentially fast and uniform for all x in any interval bounded from above.

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